Знание и бизнес, Knowledge and business

Икономика и компютърни науки
Economics and computer science

Научно списание

A scientific journal

бр. 4 / 2017

Volume: 2017, Issue: 4

Издателство „Знание и бизнес“ Варна
Publishing house „Knowledge and business“ Varna

ИКОНОМИКА И КОМПЮТЪРНИ НАУКИ

ECONOMICS AND COMPUTER SCIENCE

Научно списание

A scientific journal

Брой 4 / 2017

Volume: 2017, Issue: 4

брой 4, 2017 г.

Volume: 2017, Issue: 4

ISSN 2367-7791, Varna, Bulgaria

ЦЕНТЪР ЗА ПРОФЕСИОНАЛНО ОБУЧЕНИЕ „ЗНАНИЕ И БИЗНЕС“

Главен редактор: проф. д-р ик. н. Петко Илиев – тел: 0893 492 350

Зам. гл. редактор: доц. д-р Светлозар Стефанов – тел: 0878 362 245

Зам. гл. редактор: доц. д-р Юлиан Василев – тел: 0882 164 711

Научен секретар: доц. д-р Анастасия Кондукторова – тел: 0882 164 846

Предпечат: Юлиан Василев

9000, гр. Варна, ул. „Д-р Л. Заменхоф“ № 1

email: eknigibg@gmail.com

CONTENTS

ОПТИМИЗИРАНЕ НА ТОВАРНИТЕ ПРЕВОЗИ В МИКРОЛОГИСТИЧНИ СИСТЕМИ 7

OPTIMIZING THE FLOWS OF GOODS IN MICRO LOGISTICS SYSTEMS..17

SECOND CONFERENCE ON INNOVATIVE TEACHING METHODS – VARNA 2017. CONFERENCE SUMMARY AND EVALUATION REPORT ...18

QUADRATURE FORMULAS AND TAYLOR SERIES OF SECANT AND TANGENT 23

Dear readers, we are pleased to inform you that on the initiative of the Editorial Board of "Knowledge and Business" Ltd., "Knowledge and Business" is a licensed publisher of the scientific journal "Economics and Computer Science.".

Our thesis is that information technologies are used in all areas of human activity. That is why the scientific journal "Economics and Computer Science" is open to publications from various fields of science and practice.

We are confident that the publications in the journal contribute as enrichment of knowledge and to enhance the effectiveness of the use of information technology in business.

Sincerely: Prof. Dr. Sc. Petko Iliev

ОПТИМИЗИРАНЕ НА ТОВАРНИТЕ ПРЕВОЗИ В МИКРОЛОГИСТИЧНИ СИСТЕМИ

Радан МИРЯНОВ[1]

[1] Икономически университет – Варна
miryanov@ue-varna.bg

Резюме. В статията се разглеждат микрологистичните системи и е отправен специфичен поглед към товарните превози в тях с цел оптимизирането им. За целта са използвани методи на моделирането и математическото оптимиране. Приложимостта на разгледаните задачи и методи е нагледно показана в материала с конкретни примери. Демонстриран е и един по-различен подход – този на мрежовото планиране. При него за минимизиране на линейната форма се използва мрежов график и решението на задачата се осъществява на два етапа – първо като класическа транспортна задача, а след това – с включването на допълнително ограничение. В резултат от решаването на тези задачи се определят оптималните доставки, които следва да се отчетат при планиране на товарните превози в микрологистични системи.

Ключови думи: микрологистични системи, транспортна задача, мрежово планиране.

1. Увод

Както е известно, едно от важните направления за осигуряване на ефективно снабдяване и пласмент в дистрибуционната логистика е „моделиране и оптимизиране на логистични процеси във връзка с управление на разходите по канали за реализация, по посредници и обекти“ (Благоева и др. 2009). И в тази връзка основната цел на настоящата разработка е да се предложат възможности за оптимизиране на транспортните разходи в микрологистични системи при оценката и избора на алтернативни стратегии на транспортно обслужване между доставчици и потребители. Решаването на така поставения проблем във вид на единна екстремална задача води до сложен математически модел, практическата реализация на който е свързана с трудности от изчислителен характер, поради което е целесъобразно да бъде декомпозирана на две задачи, които следва да бъдат решавани последователно. Първоначално се решава задачата за асоцииране на потребителите от логистичните центрове съответно с базите на потребителите (т. нар. „прикрепване“). В резултат от нейното решаване, заедно с прикрепването на потребителите към базите, се определят също така и потребностите на последните от транспортирания продукт. След това следва да бъде решена задачата за прикрепване на потребителите, в това число и на

базите, към предприятията – доставчици. Именно това е логическата последователност, в която се решават тези задачи в настоящата разработка.

2. Постановка и решение на задачите

Предполага се, че са известни:

n – предприятията – потребители на транспортирания продукт;

p – доставчиците от микрологистичната система;

b_j – потребностите на *j*-тия потребител $(j = 1 \div n)$;

L_k – пропускателната способност *k*-тото логистично звено по отношение на транспортирания продукт;

c_{kj} – транспортни разходи по доставката на дадения продукт от *k*-тото логистично звено до *j*-тия потребител;

x_{kj} – търсеният обем на доставка от *k*-тото логистично звено до *j*-тия потребител.

Задачата се състои в търсенето на такива числа $x_{kj} \geq 0$ $(k = 1 \div p,\ j = 1 \div n)$, минимизиращи функцията

$$Z = \sum_{k=1}^{p} \sum_{j=1}^{n} c_{kj} x_{kj}$$

(представляваща сумарните транспортни разходи), при следните ограничителни условия:

1. Количествата от продукта, планирани за доставка от съответното логистично звено, не трябва да превишават неговата пропускателна способност

$$\sum_{j=1}^{n} x_{kj} \leq L_k\ ,\ (k = 1 \div p).$$

2. Потребностите на *j*-тия потребител от дадения продукт трябва да бъдат задоволени

$$\sum_{k=1}^{l} x_{kj} = b_j\ ,\ (j = 1 \div n).$$

По този начин се получава така наречения открит модел на класическата транспортна задача (Атанасов и Милкова 2011). В резултат от нейното решение, освен прикрепване на потребителите към съответните логистични звена от

микрологистичната система, се определят също така и обемите от продукта, които е необходимо да се доставят до всеки потребител (Милкова и Николаев 2012).

Продължаваме с решението на поставената задача с отчитане разпределението на наличните запаси от продукта в началото на плановия период. Нека с Q_k да означим тези запаси от продукта.

Всяко логистично звено условно се разделя на две части. В една от тях пропускателната способност приемаме да бъде равна на R_k , а другата

$$\left(L_k - R_k\right),\ \left(k = 1 \div p\right).$$

Въвеждаме още означенията:

x_{kj}^1 – търсеният обем доставка на продукта от логистично звено с пропускателна способност R_k на j-тия потребител;

x_{kj}^2 – търсеният обем доставка на продукта от логистично звено с пропускателна способност $\left(L_k - R_k\right)$ на j-тия потребител.

По този начин е достатъчно да се търси минимума на функцията

$$\sum_{k=1}^{p}\sum_{j=1}^{n} M\left(c_{kj}x_{kj}^1\right) + \sum_{k=1}^{p}\sum_{j=1}^{n} c_{kj}x_{kj}^2$$

при тривиалните ограничения, налагани на променливите x_{kj}^1 и x_{kj}^2 (тук M е едно достатъчно голямо положително число). В този случай следва последователно да бъдат решавани две транспортни задачи. В първия случай се решава транспортна задача за логистично звено с пропускателна способност R_k , след което за логистично звено с пропускателна способност $\left(L_k - R_k\right)$.

Ясно е, че в случай, когато $\sum_{k=1}^{p} R_k < \sum_{j=1}^{n} b_j$, намереното решение може да не бъде оптимално. Поради това следва да се постъпи по следния начин:

1. Ако $\sum_{k=1}^{p} R_k \geq \sum_{j=1}^{n} b_j$, то следва да се реши обикновена транспортна задача, при която за доставчици следва да се приемат логистични звена с пропускателна способност R_k.

2. Ако $\sum_{k=1}^{p} R_k < \sum_{j=1}^{n} b_j$, то в този случай следва да се формулира по нов начин самата задача.

Да се намери минимумът на функцията

$$\sum_{k=1}^{p}\sum_{j=1}^{n} c_{kj}\left(x_{kj}^1 + x_{kj}^2\right)$$

при следните ограничения:

$$\sum_{j=1}^{n} x_{kj}^{2} = R_k \, , \, (k = 1 \div p),$$

т.е. запасите в съответното логистично звено трябва изцяло да бъдат разпределени;

$$\sum_{j=1}^{n} x_{kj}^{1} \le (L_k - R_k),$$

т.е. количеството продукт, доставено от *k*-тото логистично звено, не трябва да превишава неговата пропускателна способност;

$$\sum_{k=1}^{p} (x_{kj}^{1} + x_{kj}^{2}) = b_j \, , \, (j = 1 \div n),$$

т.е. потребностите на *j*-тия потребител от разглеждан продукт трябва да бъдат задоволени;

$$x_{kj}^{1} \ge 0 \, , \; x_{kj}^{2} \ge 0 \, , \, (k = 1 \div p, \, j = 1 \div n),$$

т.е. доставките от продукта следва да бъдат реални.

Следва да отбележим, че допълнително формулираната задача може да бъде решена като обикновена транспортна задача, но за целта е необходим следният ред за подготовка на изходната информация. Всяко логистично звено се подразделя на две части. Въвежда се фиктивен потребител с обем на потребностите

$$b_{\phi} = \sum_{k=1}^{p} L_k - \sum_{j=1}^{n} b_j \, .$$

Транспортните разходи за доставка на единица товар от логистично звено с пропускателна способност R_k до фиктивния потребител са равни на ∞, а от логистично звено с пропускателна способност $(L_k - R_k)$ – равни на нула. Да демонстрираме решението на такава задача на база един елементарен условен пример.

Нека предположим, че са известни две логистични звена A_1 и A_2, пропускателната възможност на всяко едно от тях по отношение на превозвания продукт е съответно $L_1 = 110$ и $L_2 = 120$ единици. Запасът на първото звено е $R_1 = 70$, а на второто $R_2 = 0$ единици. В три пункта на потребление на транспортирания продукт B_1, B_2, B_3 обемите на потребление са съответно: $b_1 = 70$, $b_2 = 20$, $b_3 = 60$ единици. Транспортните разходи по доставката на продукта на логистичните звена до пунктовете на потребление (c_{kj}) са дадени в табл.1.

Таблица 1

	B_1	B_2	B_3
A_1	17	10	11
A_2	15	11	10

Тъй като в първото логистично звено има запас, подлежащ на разпределение, то е целесъобразно това звено да бъде разделено на две $A_1^{'}$ и $A_2^{''}$, тъй както е показано в табл.2.

Таблица 2

		v_j / u_i	$v_1=15$	$v_2=9$	$v_3=10$	$v_4=0$		
B_j / A_i			B_1	B_2	B_3	B_ϕ	a_i	
A_1	$A_1^{'}$	$u_1^{'}=0$	17	10	11	0	L_1-R_1=40	L_1=110
						40		
	$A_1^{''}$	$u_1^{''}=-1$	17	10	11	∞	R_1 = 70	
				20	50			
A_2		$u_2=0$	15	10	10	0	L_2 = 120	
			70		10	40		
b_j			70	20	60	80		

За приведения пример сумарната мощност на логистичните звена превишава общия обем на потребление с 80 единици. За решаването на задачата е въведен фиктивен потребител $B_\phi = 80$ ед. Приемаме за транспортни разходи по доставката на продукти от логистичните звена $A_1^{'}$ и A_2 до фиктивния потребител равни на нула, а от звеното $A_1^{''}$ - равни на ∞. За решаването на задачата е използван известният метод на потенциалите (табл. 2).

Както се вижда от полученото решение, логистичното звено A_1 е изцяло прикрепено към реалните потребители. Това означава, че запасите в логистичното звено A_1 са разпределени. Намереното решение определя количеството от продукта, което е необходимо да се достави от логистичните звена до потребителите. Освен това, сумата от доставките на продукта от двете логистични звена A_1 и A_2 до всички реални потребители определя потребностите на звената от дадения продукт.

По-сложна задача възниква при прикрепване на потребители към логистични звена или предприятия – производители, в случай когато се транспортират опаковъчни единици (пакетирани неделими партиди от товара), тъй като в този случай на променливите величини се налага допълнително изискване за целочисленост.

Първоначално се разглежда задачата за оптимално прикрепване на потребители към предприятия – доставчици за случая на дискретно изменящи се бройки от опаковъчни единици. За целта въвеждаме следните означения:

b_j – количеството на опаковъчните единици, които е необходимо да се доставят на *j*-тия потребител $\left(j=1\div n\right)$ (b_j се задава с отчитане на самите опаковъчни единици);

a_i - количеството на продукта, подлежащо на доставяне от *i* –тото предприятие – доставчик $\left(i=1\div m\right)$;

c_{ij} – транспортни разходи по доставка на продукти от *i*-тия до *j*-тия пункт;

G – големина на опаковъчната единица;

x_{ij} – търсеният обем на доставка от *i*-тия доставчик до *j*-тия потребител.

Необходимо е да се намерят неотрицателни стойности на променливите, минимизиращи линейната форма

$$F=\sum_{j=1}^{m}\sum_{j=1}^{n}c_{ij}x_{ij}$$

при ограничения:

1. Потребностите на *j*-тия потребител трябва да бъдат задоволени

$$\sum_{i=1}^{m}x_{ij}=b_j\ ,\ \left(j=1\div n\right).$$

2. Количеството на продукта, доставено от *i*-тия доставчик не трябва да превишава възможните мощности на предприятието – доставчик

$$\sum_{j=1}^{n}x_{ij}\le b_i\ ,\ \left(i=1\div m\right).$$

3. Обемът на доставката трябва да бъде кратен на опаковъчните единици

$$x_{kj}\in\{kG\},\ \left(i=1\div m,\ j=1\div n\right),$$

където *k* е цяло неотрицателно число.

Математическото решение на такава задача не представлява особена трудност. Необходимо е да се разделят $a_i\left(i=1\div m\right)$ и $b_j\left(j=1\div n\right)$ на големината на опаковъчната единица (предполага се, че изходната информация е подготвена така, че a_i и b_j да са кратни на опаковъчните единици) и да се реши съответната транспортна задача. Както е известно, алгоритъмът за решаване на транспортната задача при направените предписания води до намиране на целочислени решения (Желязкова 2017).

3. Метод на мрежовото планиране

За решаването на подобни задачи може да се постъпи по следния начин. Разделяме потребностите на всеки потребител на няколко части, така че всяка една от тях да бъде равна на една от опаковъчните единици. Следователно, по този начин може да се разглеждат потребителите с потребности $G_\mu \ (\mu = 1 \div r)$, където G_μ е една от опаковъчните единици. Ако с $c_{i\mu}$ означим транспортните разходи по доставка на продукта от *i*-тия пункт в μ-тия, то задачата може да се формулира по следния начин:

Да се намерят такива неотрицателни стойности на променливите $x_{i\mu} \ (i = 1 \div n, \mu = 1 \div r)$ минимизиращи функцията:

$$F = \sum_{i=1}^{m} \sum_{\mu=1}^{r} c_{i\mu} x_{i\mu}$$

при следните ограничителни условия:

1. $\sum_{i=1}^{m} x_{i\mu} = G_\mu , (\mu = 1 \div r),$

2. $\sum_{\mu=1}^{r} x_{i\mu} \le a_i , (i = 1 \div m),$

3. $x_{i\mu} \in \{0, G_\mu\}, \ \mu = 1 \div r .$

За решаване на формулираната задача може да бъде използван методът на мрежовото планиране. Приемаме за връх на мрежовия график точка, на която съответства набор от *m* числа (всяко от тези числа представлява целочислена комбинация от зададените опаковъчни единици и не превишава съответните наличности a_i), т.е техният брой да съответства на този на потребителите.

Първото число показва какво количество от продукта следва да се транспортира от първия доставчик до всички потребители, включително до μ-тия, второто – от втория доставчик и т.н. За дължина на дъгите се приемат величините $c_{i\mu}$ - транспортните разходи.

Построяването на мрежовия график започва с върха 0_1. Всяка излизаща от него дъга определя транспортните разходи по превоза на продукта от *i*-тия доставчик до първия потребител. В новополучените върхове наборите от числа имат вида $(0,0,..., G_\mu, 0,..., 0)$. От всеки от получените върхове излизат дъги, задаващи транспортните разходи по превоза на продукта от *i*-тия доставчик до втория потребител.

Два върха от мрежовия график са съединени с дъги в този и само в този случай, ако всеки две от числата на набора съвпадат, освен едно, което се различава от

числата от предходния набор само по големината на опаковъчната единица. Построяването продължава дотогава, докато бъдат обхванати всички потребители.

Върховете, съответстващи на последния потребител, се съединяват с дъги в точка 0_2. Дължините на тези дъги се приемат равни на нула.

В общия случай наборът от числата при всеки от върховете има вида:

$$\left(K_{1\mu}, K_{2\mu},..., K_{m\mu}\right),$$

където $K_{i\mu} = k_{i\mu}G_{\mu}$, $k_{i\mu}$ - цяло неотрицателно число.

По такъв начин се построява мрежов график, съответстващ на поставената задача. Във връзка с това, че дължината на дъгите в такъв мрежов график представят по същество транспортните разходи, то най-краткият път между точките 0_1 и 0_2 задава оптималния план. Методът за намиране на най-икономичния маршрут е известен (Милкова и Михайлов 2016).

Да демонстрираме построяването на мрежовия график посредством следния пример. Налице са три доставчици на продукт, съответно $a_1 = 10$, $a_2 = 13$ и $a_3 = 17$ единици и четири потребители с обем на потребление $b_1 = 4$, $b_2 = 4$, $b_3 = 4$, $b_4 = 9$. Известни са и опаковъчните единици $G_1 = 4$, $G_2 = 9$ и транспортните разходи, които са записани в горните десни ъгли на табл.3.

Таблица 3

A_i \ B_j	u_i \ v_j	$v_1 = 13$	$v_2 = 11$	$v_3 = 14$	$v_4 = 4$	$v_5 = -2$	
		B_1	B_2	B_3	B_4	B_{ϕ}	a_i
A_1	$u_1 = 0$	13 | 0	18	14 | 0	4 | 9	0	10
A_2	$u_2 = +1$	12 | 4	10 | 4	24	24	0	13
A_3	$u_3 = -2$	10	25	16 | 4	18	0 | 13	17
b_j		4	4	4	9	19	

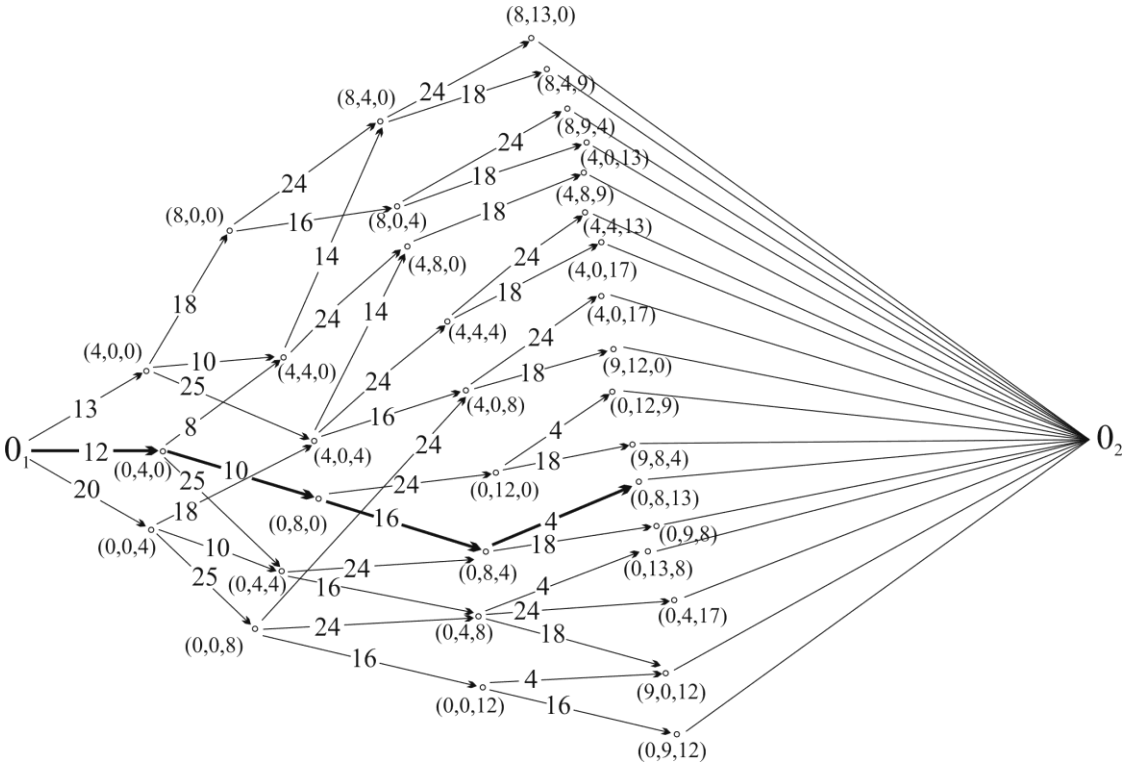
Построяваме мрежовия график (фиг.1) и търсим най-краткия път между точките 0_1 и 0_2. На този най-кратък път съответства оптимален план, определен в таблица 3.

В случаите, когато задачата е с голяма размерност, може да се използва приближен метод.

Решението на задачата може да бъде осъществено на два етапа. На първия етап се решава задачата за намиране на минимума на функцията F при първите две

ограничителни условия, т.е. класическата транспортна задача. На втория етап следва да се коригира полученото решение с цел отчитане и на третото ограничение.

След първия етап за прикрепени се приемат обемите, кратни на опаковъчните единици. На втория етап предмет на разглеждане са останалите обеми. За тяхното прикрепване следва последователно да бъдат решавани редица транспортни задачи. В началото се решава задачата за прикрепване на потребителите, в които са останали обеми, кратни на опаковъчните единици. След това следва да бъде решена задачата за прикрепване, при която се разглеждат потребители, имащи обеми, кратни на следващата величина доставка и т.н.



Фиг.1

4. Извод

В резултат от решаването на тези задачи към логистичните звена са прикрепени потребителите и определени оптималните размери на партидата на доставка за всеки от тях. Следователно доставката от логистичните звена е необходимо да бъде планирана по такъв начин, че размерите на партидите при окомплектоването на товарните превози да съответстват на получените значения на товарните единици.

Литература/Literature

Atanasov, B. and Milkova, T., 2011. Quantitative Methods in Logistics. - Varna: Univ. Ed. "Science and Economics" (in Bulgarian).

Blagoeva, S. and others, 2009. Economic Logistics. Varna: Univ. Ed. "Science and Economics" (in Bulgarian).

Zhelyazkova, D., 2017. Transport logistics. Varna: Univ. Ed. "Science and Economics" (in Bulgarian).

Milkova, T. and Mihailov, D., 2016. Investigation of Operations. - Varna: Univ. Ed. "Science and Economics" (in Bulgarian).

Milkova, T. and Nikolaev, R., 2012. Models for optimal attachment of users to suppliers. Years. Icon. N. - Varna, 84 (in Bulgarian).

OPTIMIZING THE FLOWS OF GOODS IN MICRO LOGISTICS SYSTEMS

Radan MIRYANOV[1]

[1] University of Economics, Varna, Bulgaria
miryanov@ue-varna.bg

Abstract. In the present paper micro logistics systems are observed with a specific glimpse at the flows of goods in them, aiming their optimization. For this reason some aspects of methodology of modelling and mathematical optimization are used. The practical implications of the problems and methods concerned are demonstrated with particular examples. A different approach is also revealed – the so called network planning. It uses network graphics for minimizing the linear form and the solution is split into two – first as a classical transportation problem, then with another constraint added. Finally, as a result, the optimal quantities are determined and they must be taken into consideration when planning the flows of goods in micro logistics systems.

Key words: micro logistics systems, transportation problem, network planning.

SECOND CONFERENCE ON INNOVATIVE TEACHING METHODS – VARNA 2017. CONFERENCE SUMMARY AND EVALUATION REPORT

Julian VASILEV[1], Miglena STOYANOVA[2]

[1] University of Economics, Varna, Bulgaria
vasilev@ue-varna.bg

[2] University of Economics, Varna, Bulgaria
m_stoyanova@ue-varna.bg

Abstract. The purpose of this paper is to present the process of organizing, conducting and evaluating a conference. The conference is Second conference on innovative teaching methods (http://conf.000webhostapp.com). The conference was organized by University of Economics Varna within the DIMBI project (www.dimbi.eu) in Swiss-Belhotel Dimyat Varna. Data for this paper is collected through an online questionnaire with 20 questions using Google forms. Using descriptive statistics and manual text mining techniques the conference evaluation report is created. The conference summary report is given in sections 1 and 2. The conference evaluation report is given in section 3.

Key words: DIMBI project, conference evaluation

1. Preparation for the conference

The 2nd conference on innovative teaching methods (http://conf.000webhostapp.com) was the second multiplier event within the DIMBI project (www.dimbi.eu). The conference was organized by University of Economics Varna. It was held in Swiss-Belhotel Dimyat (http://www.swiss-belhotel.com/bg-bg/swiss-belhotel-varna), Perla hall, on 28 and 29 June 2017. A paper template and a flyer are created and uploaded on the conference web site.

The total number of participants was 71. They were 24 foreign participants and 47 local participants (from Bulgaria). The initial project goal was to have 10 foreign participants and 40 local participants. This goal is achieved. The local participants from universities in Varna were 32. The local participants from other Bulgarian towns (Sofia, Svishtov and Shumen) were 15. Foreign participants were from Poland (Polkowice, Wroclaw, Szczecin and Katowice), India (Navi Mumbai and Ahmedabad), Romania (Sibiu and Pitesti), Ukraine (Dnipro) and Serbia (Novi Sad). The book with conference proceedings contains 39 papers written by 67 people. Some papers are written by several authors. The book with conference proceedings is 254 pages so far. The average paper length is 6.5 pages. Several papers are still expected – to pass the review process and plagiarism check and to be added to the conference proceedings.

The conference organizers received 41 papers. 2 papers are rejected. 6 papers are returned for revising due to bad style or plagiarism. The authors of these 6 papers have revised them. The papers passed again the review process and plagiarism check and they were included in the conference proceedings. 33 papers are accepted without revisions. Three people were engaged with plagiarism check and one person with the review process.

The conference proceedings are visible on the conference web site – they are published online with an ISBN. The book with conference proceedings is indexed in RePEc (http://econpapers.repec.org/bookchap/vrndimbip/23.htm). According to Christian Zimmermann (web master of RePEc) the page views of the conference proceedings (http://econpapers.repec.org/bookchap/vrndimbip/) in RePEc during June 2017 are 64. The file downloads from RePEc during June 2017 are 25. (See http://logec.repec.org/scripts/seritemstat.pf?h=repec:vrn:dimbip).

2. Conducting the conference

From all 71 participants: 39 people attended the conference physically, 3 people participated with posters during a poster session, 29 people participated remotely. 4 people attended the conference without participating with a paper. Members of the DIMBI project, who took part in the conference, are 3 people from University of Economics Varna, 1 person from Jan Wyzykowsky University in Polkowice and 1 person from University of Economics Wroclaw.

During the conference several coffee breaks were provided. Lunch and dinner were also provided for the conference participants. Three twin rooms were also used by conference participants coming from Sofia, Polkowice and Navi Mumbai.

3. Conference evaluation

The conference evaluation is conducted through an anonymous online questionnaire, send to 59 participants (42 local participant and 17 foreign participants). 28 answers are received. The response rate is 47%.

The conference survey was conducted after the conference (29 June – 03 July 2017). The received answers are converted into an XLSX file and after that – to a SAV file for PSPP or SPSS. Coding of variables is done. Using descriptive statistics several tables with aggregated data are created. Conference satisfaction is measured with questions Q1-Q12 (table 1).

Table 1.

Results of the questions concerning conference satisfaction (Q1-Q12)

Questions	Given answers (in %)				
	1	2	3	4	5
	Strongly disagree	…	…	…	Strongly agree
Q1. I am satisfied with the conference as a whole	0.0	0.0	3.6	25.0	71.4
Q2. I had time to look at the conference proceedings before the conference	0.0	3.6	3.6	28.6	64.3
Q3. Some papers in the conference proceedings attracted my attention. I plan to contact the authors of these papers.	0.0	0.0	10.7	35.7	53.6
Q4. The event met my expectations.	0.0	0.0	3.6	28.6	67.9
Q5. The conference program was well organized.	0.0	0.0	3.6	17.9	78.6
Q6. The event was well prepared.	0.0	0.0	3.6	21.4	75.0
Q7. The organisers were helpful.	0.0	0.0	0.0	14.3	85.7
Q8. The atmosphere was friendly.	0.0	0.0	3.6	17.9	78.6
Q9. We had a chance for remote participation in the conference.	0.0	0.0	14.3	25.0	60.7
Q10. I extended my knowledge in innovative teaching methods.	0.0	0.0	3.6	28.6	67.9
Q11. The conference web site was useful.	0.0	0.0	10.7	14.3	75.0
Q12. The catering during the conference was OK.	0.0	0.0	3.6	10.7	85.7

Source: Own calculations in PSPP

71.4% of all respondents are satisfied with the conference as a whole. The conference satisfaction in different dimensions is measured with questions Q2 to Q12. 64.3% of the respondents had time to look at the conference proceedings before the conference. 53.6% of the respondents plan to contact the authors of some of the read papers because they attracted their attention. 67.9% of the respondents strongly agree that the event met their expectations. 78.6% of the respondents strongly agree that the conference program was well organized. 75% of the respondents strongly agree that the conference was well prepared. 85.7% of the respondents strongly agree that the organizers were helpful. 78.6% of the respondents strongly agree that the atmosphere was friendly. 60.7% of the respondents strongly agree that they had a chance for remote participation. 67.9% of the respondents strongly agree that they extended their knowledge in innovative teaching methods. 75% of the respondents strongly agree that the conference web site was useful. 85.7% of the respondents strongly agree that the catering during the conference was OK.

Respondents were asked the possible reasons for attending the conference. They had a chance to mark more than one reason (table 2).

Table 2.

Results of the questions concerning the reasons for attending the conference (Q13.1-Q13.5)

Questions	Given answers (in %)	
	The option is checked	The option is not checked
Q13.1. The conference proceedings are indexed in RePEc	71.4	28.6
Q13.2. There is no conference fee	50.0	50.0
Q13.3. The conference is in Varna	60.7	39.3
Q13.4. I have the opportunity to meet new colleagues	67.9	32.1
Q13.5. I have the opportunity to exchange ideas	82.1	17.9

The strongest reason for attending the conference was the opportunity to exchange ideas. This reason is followed by the fact that the conference proceedings are indexed in RePEc. Almost 2/3 of the respondents attended the conference because of meeting new colleagues. The absence of a conference fee was not a strong reason for attending it.

Since respondents had the chance to mark several reasons, the number of reasons is aggregated by the number of respondents (table 3).

Table 3.

Results of the questions concerning the reasons for attending the conference

Count of reasons to attend the conference	Number of respondents
1 (one reason was marked)	6
2 (two reasons were marked)	3
3 (three reasons were marked)	4
4 (four reasons were marked)	6
5 (five reasons were marked)	9
Total	28

Q14 is an open text question “If you have recommendations to the conference organizers, please, write them here”. Three people said that they do not have any recommendations. Five people wrote: “should be continued”, “Well done!”, “Keep doing the great job!!!”, “When will be the next conference?”, “I am interested in the organizers' research”. 20 people did not give any recommendations.

Q15 is oriented to checking if the participant is “local” (from Bulgaria) or “foreign” (from abroad). 82.1% of the respondents are “local”. 17.9% of the respondents are “foreign”. The ratio between local and foreign participants in the respondents (n=28) is 23/5 = 4.60. The ratio between local and foreign participants in the population (N=71) is 47/24 = 1.96. Local participants were more active in filling in the survey. One of the possible reasons for this discrepancy was the fact that several papers from foreign participants were sent to the organizing committee after finishing the conference evaluation survey period (29 June – 03 July 2017).

Q16 is oriented to the preferences of people to participate in a conference – to attend physically or to participate remotely. 82.1% of the respondents prefer „to attend a conference“. 17.9% of the respondents prefer „to participate remotely“.

Q17 is about the review process and plagiarism check. 67.9% of the respondents said that their paper was „accepted without revisions“. 21.4% of the respondents said that their paper was „accepted with minor revisions“. 10.7% of the respondents did not answer this question.

Q18 is “How did you hear about the conference?” – a question with multiple choice answers. The aggregated results are given in table 4.

Table 4.

Aggregated results on question Q18 about the conference invitation

Types of conference invitation	Percent
Conference web site; e-mail from colleagues	3.6
E-mail from colleagues	39.3
Personal invitation	42.9
Personal invitation, conference flyer, conference web site	3.6
Personal invitation, conference web site	3.6
Personal invitation, e-mail from colleagues	3.6
No answer	3.6
Total	100.0

Informing researchers about the conference was done using several methods. The most useful one was “personal invitation”.

Q19 is “What was the most useful aspect of this conference?”. 57.1% of the respondents gave an answer to this question. 42.9% of the respondents did not answer it. Several respondents wrote “meeting colleagues”, “exchanging ideas”, “exchanging useful ideas”, „knowledge exchange“, “internationalization”. Oher participants answered “collaboration with participants”, “international participants”, “relationship between academic participants and different views in scientific area”, „KISS (keep it simple and small) approach, thus – very useful“, „new experience for me presenting in English“.

Q20 is about gender. 50.0% of the respondents are male, 42.9% – female, 7.1% – did not answer this question.

4. Conclusion

The second conference on innovative teaching methods (ITM 2017) was well prepared and well organized according to the opinions of conference participants. The conference met their expectations. Most of them strongly agree that they are satisfied from the event.

The results in this study are based mainly on descriptive statistics using the answers of a conference evaluation survey. Future research may focus on creating cross-tables using the same answers.

The underlying structure of the conference satisfaction scale may be measured by factor analysis (principal component analysis – PCA). The PCA is not suitable for our dataset because there should be 150+ cases. But PCA may be useful for bigger conferences.

QUADRATURE FORMULAS AND TAYLOR SERIES OF SECANT AND TANGENT

Yuri DIMITROV[1], Radan MIRYANOV[2], Venelin TODOROV[3]

[1] University of Forestry, Sofia, Bulgaria
yuri.dimitrov@ltu.bg

[2] University of Economics, Varna, Bulgaria
miryanov@ue-varna.bg

[3] Bulgarian Academy of Sciences, IICT, Department of Parallel Algorithms, Sofia, Bulgaria
venelin@parallel.bas.bg

Abstract. Second-order quadrature formulas and their fourth-order expansions are derived from the Taylor series of the secant and tangent functions. The errors of the approximations are compared to the error of the midpoint approximation. Third and fourth order quadrature formulas are constructed as linear combinations of the second-order approximations and the trapezoidal approximation.

Key words: Quadrature formula, Fourier transform, generating function, Euler-Mclaurin formula.

1. Introduction

Numerical quadrature is a term used for numerical integration in one dimension. Numerical integration in more than one dimension is usually called cubature. There is a broad class of computational algorithms for numerical integration. The Newton-Cotes quadrature formulas (1.1) are a group of formulas for numerical integration where the integrand is evaluated at equally spaced points. Let $h = (b-a)/n$.

$$h\sum_{k=0}^{n} w_k f(b-kh) \approx \int_a^b f(x)dx. \tag{1.1}$$

The two most popular approximations for the definite integral are the trapezoidal approximation (1.2) and the midpoint approximation (1.3).

$$A_T^h f(x) = \frac{h}{2}\left(f(a) + 2\sum_{k=1}^{n-1} f(a+kh) + f(b)\right) = \int_a^b f(x)dx + O(h^2), \tag{1.2}$$

$$A_M^h f(x) = h\sum_{k=1}^{n-1} f(a+(k+1/2)h) = \int_a^b f(x)dx + O(h^2). \quad (1.3)$$

The midpoint approximation and the trapezoidal approximation for the definite integral are Newton-Cotes quadrature formulas with accuracy $O(h^2)$. The trapezoidal and the midpoint approximations are constructed by dividing the interval $[a,b]$ to subintervals of length h and interpolating the integrand function by Lagrange polynomials of degree zero and one. Another important Newton-Cotes quadrature formula is the Simpson's approximation. The Simpson's approximation has fourth-order accuracy and is obtained by interpolating the function by a second degree Lagrange polynomial. Denote by $E_T^h f(x)$ and $E_M^h f(x)$ the errors of the trapezoidal and the midpoint approximations.

$$E_T^h f(x) = A_T^h f(x) - \int_a^b f(x)dx + O(h^2), E_M^h f(x) = A_M^h f(x) - \int_a^b f(x)dx + O(h^2).$$

From the Euler-Mclaurin formula the error of the trapezoidal approximation on the interval $[a,b]$ is expressed as

$$E_T^h f(x) = \sum_{k=1}^{m} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)h^{2k} + R_{2m}^T(h)h^{2m}, \quad (1.4)$$

$$R_{2m}^T(h) = \frac{1}{(2m)!}\int_a^b \tilde{B}_{2m}\left(a - \frac{t}{h}\right)f^{(2m)}(t)dt,$$

where $\tilde{B}_n(x) = B_n(\{x\}) = B_n(x - \lfloor x \rfloor)$ is the Bernoulli 1-periodic function and $B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k$ is the Bernoulli polynomial of degree n. The error of the midpoint approximation satisfies (Kouba 2013)

$$E_M^h f(x) = \sum_{k=1}^{m} \frac{(2^{1-2k}-1)B_{2k}}{(2k)!}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)h^{2k} + R_{2m}^M(h)h^{2m}, \quad (1.5)$$

$$R_{2m}^M(h) = \frac{1}{(2m)!}\int_a^b \tilde{B}_{2m}\left(\frac{a+b}{2} - \frac{t}{h}\right)f^{(2m)}(t)dt.$$

The terms $R_{2m}^T(h)$ and $R_{2m}^M(h)$ satisfy $\lim_{h\to 0} R_{2m}^T(h) = \lim_{h\to 0} R_{2m}^M(h) = 0$, when the function $f(x) \in C^{2m}[a,b]$. From (4), (5) and $m = 2$ we obtain the fourth-order expansion formulas (2.1), (2.14) of the trapezoidal and midpoint rules. The midpoint approximation is around two times more accurate than the trapezoidal approximation. The Simpson's approximation is constructed as a linear combination of the midpoint approximation and the trapezoidal approximation with weights $1/3$ and $2/3$. With this choice of the weights the second-order terms of expansion formulas (1.4) and (1.5) are cancelled and the Simpson's approximation has an accuracy $O(h^4)$.

The Gaussian quadrature formulas $\sum_{k=1}^{n} w_k f(x_k)$ are approximations for the definite integral where the integrand is evaluated at unequally spaced points (Gil et al. 2007). The Gaussian quadrature formulas are constructed to yield an exact result for the polynomials of degree $2n-1$ by using suitable values of the nodes x_k and weights w_k. The accuracy of the Newton-Cotes and the Gaussian quadrature formulas is lower when the function $f(x)$ has singularities.

The Monte Carlo methods for numerical integration are computational algorithms where the nodes are chosen as pseudo-random or quasi-random numbers in the area of integration, the unit hypercube (Dimov 2008). The Monte-Carlo and quasi-Monte Carlo methods are suitable for approximation of multidimensional integrals, since the convergence of the methods is independent of the dimension (Atanassov & Dimov 1999; Georgieva 2009; Todorov & Dimov 2016). The generating function of quadrature formula (1.1) is defined as

$$G(x) = \sum_{k=0}^{\infty} w_k x^k.$$

The Riemann sum approximation for the definite integral is related to the trapezoidal approximation and has a generating function $1/(1-x)$. The Riemann sum approximation for the fractional integral of order α has a generating function $Li_{1-\alpha}(x)$, where $Li_{1-\alpha}(x)$, is the polylogarithm function of order $1-\alpha$. The midpoint approximation is a shifted approximation with shift parameter $h/2$. The midpoint approximation has a generating function $\sqrt{x}/(1-x)$. Methods for approximation of fractional integrals and derivatives based on the Fourier transform and the generating function of the approximation have been studied by (Chen & Deng 2016; Ding & Li 2016; Lubich 1986; Tuan & Gorenflo 1995). In (Dimitrov 2016) we use the series expansion formula of the polylogarithm function $Li_{1-\alpha}(e^x)$, to derive the expansion formula of the trapezoidal approximation for the fractional integral of order α. In the present paper we construct second, third and fourth order approximations for the definite integral with generating functions $G_1(x) = \pi\sec(\pi\sqrt{x}/2)/4$ and $G_2(x) = \pi\tan(\pi\sqrt{x}/2)/(4\sqrt{x})$. The two generating functions $G_1(x)$ and $G_2(x)$ are positive, increasing functions and have a vertical asymptote at the point $x = 1$. The graphs of the functions $G_1(x), G_2(x)$ and $1/(1-x)$ are given in Figure 1.

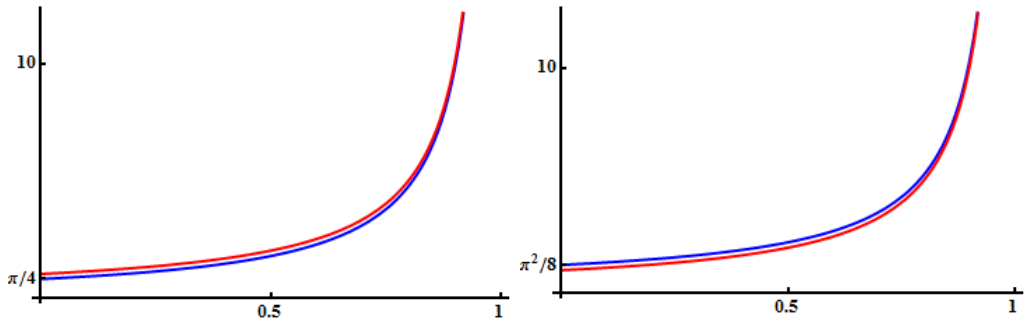


Figure 1. Graphs of the generating functions $G_1(x) = \pi\sec(\pi\sqrt{x}/2)/4$ *(left, blue)* and $G_2(x) = \pi\tan(\pi\sqrt{x}/2)/(4\sqrt{x})$ *(right, blue)* and the function $1/(1-x)$ *(red).*

The tangent and secant functions have Mclaurin series expansions

$$\sec x = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k)!} x^{2k}, \tag{1.6}$$

$$\tan x = \sum_{k=0}^{\infty} \frac{(-4)^{k+1}(1-4^{k+1})B_{2k+2}}{(2k+2)!} x^{2k+1}. \tag{1.7}$$

The radius of convergence of power series (1.6) and (1.7) is $|x| < \pi/2$. The first eight elements of the sequences of Euler and Bernoulli numbers are $1,0,-1,0,5,0,-61,0,\cdots$ and $1,-1/2,1/6,0,-1/30,0,1/42,0,\cdots$. The Euler and Bernoulli numbers with an odd index are equal to zero. The sequence of Euler numbers $\{E_{2n}\}$ is an alternating sequence of integers, and $\{B_{2n}\}$ is an alternating sequence of rational numbers. The Bernoulli and Euler numbers satisfy the asymptotic relation $4^n(1-4^n)B_{2n}/E_{2n} \sim \pi/2$. Efficient methods for computation of the Euler numbers, Bernoulli numbers and the series expansions of the secant and tangent functions are discussed by (Atkinson 1986; Brent & Harvey 2013; Knuth & Buckholtz 1967). The values of the Euler numbers, Bernoulli numbers and the zeta function are included in the modern software packages for scientific computing. Denote by $\bar{E}_k$ and $\bar{B}_k$ the coefficients of the Mclaurin series expansions of the generating functions $G_1(x)$ and $G_2(x)$.

$$\bar{E}_k = \frac{|E_{2k}|}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1}, \qquad \bar{B}_k = \frac{(4^{k+1}-1)\pi^{2k+2}|B_{2k+2}|}{(2k+2)!}.$$

In section 2 and section 3 we obtain the fourth-order expansion formulas of the approximations with generating functions $G_1(x)$ and $G_2(x)$ and the second-order quadrature formulas

$$\frac{h}{2}\left(\frac{\pi-1}{2}f(a)+\sum_{k=1}^{n-1}\bar{E}_k f(a+kh)+f(b)\right)=\int_a^b f(x)dx+O(h^2),$$

$$\frac{h}{2}\left(\frac{\pi^2-6}{4}f(a)+\sum_{k=1}^{n-1}\bar{B}_k f(a+kh)+f(b)\right)=\int_a^b f(x)dx+O(h^2).$$

The substitution $\bar{f}(x) = f(a+b-x)$ yields the second-order quadrature formulas (2.12) and (3.6). In Theorem 2 and Theorem 4 we derive the conditions for the integrand function, such that the errors of second-order approximations are smaller than the error of the midpoint approximation. In section 4 we obtain third and fourth order approximations for the definite integral as linear combinations of approximations (2.12), (2.12), (3.6) and (3.7) and the trapezoidal approximation

2. Second-order quadrature formulas with generating function $G_1(x)$

In this section we derive the fourth-order expansion formula (2.9) of the approximation for the definite integral with generating function $G_1(x) = \pi\sec(\pi\sqrt{x}/2)/4$, and the second-order quadrature formulas (2.12) and (2.13). The generating function of an approximation is directly related to the result of the Fourier transform of the approximation. The Fourier transform of the function f(x) is defined as

$$\mathcal{F}[f(x)](\omega) = \hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}f(t)dt.$$

Denote by $f^{(-1)}(x)$ the definite integral of the function $f(x)$,

$$f^{(-1)}(x) = \int_{-\infty}^{x} f(t)dt,$$

and by $f^{(-n)}(x)$ the n-fold integral on the interval $(-\infty, x]$. When the function $f(x)$ is defined on the interval $[a, b]$ we extend the function to $(-\infty, b]$ by setting $f(x) = 0$ on $(-\infty, a]$. The Fourier transform has properties

$$\mathcal{F}[f(x-a)](\omega) = e^{i\omega a}\hat{f}(\omega), \qquad \mathcal{F}\left[f^{(n)}(x)\right](\omega) = (-i\omega)^n \hat{f}(\omega).$$

The trapezoidal approximation has a fourth-order expansion formula (1.4)

$$A_T^h f(x) = \int_a^b y(t)dt + \frac{1}{12}(f'(b) - f'(a))h^2 + O(h^4). \tag{2.1}$$

In (Dimitrov 2016) we use Fourier transform to prove the Euler-Mclaurin formula and we derive the asymptotic expansion formula of the trapezoidal approximation for the fractional integral. In Lemma 1 and Theorem 1 we derive the fourth-order expansion formula of the approximation for the definite integral with generating function $G_1(x)$. From the Mclaurin series of secant (1.6) we obtain

$$G_1(x) = \frac{\pi}{4}\sec\left(\frac{\pi\sqrt{x}}{2}\right) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n E_{2n}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n+1} x^n. \tag{2.2}$$

Denote by $H_1(x)$ the function

$$H_1(x) = xG_1(e^{-x}) = \frac{\pi}{4}x\sec\left(\frac{\pi e^{-x/2}}{2}\right).$$

The function $H_1(x)$ has a first derivative

$$H'_1(x) = \frac{\pi}{4}x\sec\left(\frac{\pi e^{-x/2}}{2}\right) - \frac{\pi^2}{16}x\, e^{-x/2}\sec\left(\frac{\pi e^{-x/2}}{2}\right)\tan\left(\frac{\pi e^{-x/2}}{2}\right). \tag{2.3}$$

Now we use L’Hospital’s rule to compute the values of the functions $H_1(x)$ and $H'_1(x)$ at x = 0. Substitute

$$y = \frac{\pi}{2}e^{-\frac{x}{2}}, \qquad x = -2\ln\left(\frac{2y}{\pi}\right). \tag{2.4}$$

Then

$$H_1(0) = \lim_{x\to 0^+}\frac{\pi x}{4}\sec\left(\frac{\pi e^{-x/2}}{2}\right) = \frac{\pi}{4}\lim_{y\to\frac{\pi}{2}^-}\frac{-2\ln(\frac{2y}{\pi})}{\cos y} = \frac{\pi}{2}\lim_{y\to\frac{\pi}{2}^-}\frac{1}{y\sin y} = 1.$$

From (2.3) and (2.4)

$$H'_1(x) = \frac{\pi}{4}(\sec y + y\ln(2y/\pi)\tan y) = \frac{\pi}{4}\left(\frac{\cos y + y\ln(2y/\pi)\sin y}{\cos^2 y}\right),$$

$$H'_1(0) = \frac{\pi}{4}\lim_{y\to\frac{\pi}{2}^-}\frac{\cos y + y\ln(2y/\pi)\sin y}{\cos^2 y} = \frac{\pi}{4}\lim_{y\to\frac{\pi}{2}^-}\frac{(y\cos y + \sin y)\ln(2y/\pi)}{-\sin 2y},$$

$$H'_1(0) = \frac{\pi}{4}\lim_{y\to\frac{\pi}{2}^-}\frac{\cos y + 2\cos y\ln(2y/\pi) + \sin y\,/y - y\ln(2y/\pi)\sin y}{-2\cos 2y} = \frac{1}{4}.$$

Similarly, we obtain the values of the second and third derivatives of the function $H_1(x)$ at the point $x = 0$.

$$H_1''(0) = \frac{2+\pi^2}{48}, \quad H_1'''(0) = -\frac{\pi^2}{64}.$$

The function $H_1(x)$ has a fourth-order Mclaurin series expansion

$$H_1(x) = \frac{\pi x}{4}\sec\left(\frac{\pi e^{-x/2}}{2}\right) = 1 + \frac{1}{4}x + \frac{2+\pi^2}{96}x^2 - \frac{\pi}{384}x^3 + O(x^4). \tag{2.5}$$

Denote by $A_h^S f(x)$ the quadrature formula with generating function $G_1(x)$.

$$A_h^S f(x) = \frac{h}{2}\sum_{k=0}^{n-1}\bar{E}_k\, f(b-kh) = \frac{h}{2}\sum_{k=0}^{n-1}\frac{(-1)^k E_{2k}}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1} f(b-kh). \tag{2.6}$$

Approximation (2.6) has weights $\bar{E}_k/2$ which are the coefficients of the Mclaurin series of the generating function $G_1(x)$. In Theorem 1 we use Fourier transform to show that the coefficients of the fourth-order expansion formula of (2.6) at the endpoint $x = b$ are the coefficients of the Mclaurin series of the function $H_1(x)$. The method is used in (Chen & Deng 2016; Dimitrov 2016; Ding & Li 2016; Lubich 1986; Tadjeran et al. 2006; Tuan & Gorenflo 1995) for construction of approximations of fractional integrals and derivatives. In (Dimitrov 2014) we discuss the conditions for the function $f(x)$ at the point $x = a$ for the Grünwald approximation of the Caputo derivative. The n-th order approximation for the definite integral at the point $x = b$ requires that the value of the integrand function and its derivatives of order $1,2,\cdots,n-1$ at the point $x = a$ are equal to zero.

Lemma 1 *Let* $f \in C^4[a,b]$ *and* $f^{(n)}(0) = 0$ *for* $n = 1,2,3,\ldots$ *Then*

$$A_h^S f(x) = \int_a^b f(x)dx + \frac{1}{4}f(b)h + \frac{2+\pi^2}{96}f'(b)h^2 - \frac{\pi^2}{384}f''(b)h^3 + O(h^4). \tag{2.7}$$

Proof. By applying Fourier transform to $A_h^S f(x)$ we obtain

$$\mathcal{F}[\,A_h^S f(x)](\omega) = \frac{h}{2}\sum_{k=0}^{n-1}\bar{E}_k\, e^{i\omega kh}\hat{f}(\omega) = \frac{\pi h}{4}\sum_{k=0}^{n-1}\bar{E}_k\,(e^{i\omega h})^k\hat{f}(\omega).$$

From (2.2) and $n \to \infty$ we obtain

$$\mathcal{F}[\,A_0^S f(x)](\omega) = \frac{\pi h}{4}\sec\left(\frac{\pi}{2}e^{i\omega h/2}\right)\hat{f}(\omega) = -\frac{\pi}{4i\omega}(-i\omega h)\sec\left(\frac{\pi}{2}e^{\frac{i\omega h}{2}}\right)\hat{f}(\omega).$$

Substitute $x = -i\omega h$ in (2.5)

$$\mathcal{F}[\,A_h^S f(x)](\omega) = \left((-i\omega)^{-1} + \frac{1}{4}h + \frac{2+\pi^2}{96}(-i\omega)h^2 - \frac{\pi^2}{384}(-i\omega)^2h^3\right)\hat{f}(\omega) + O(h^4).$$

By applying inverse Fourier transform we obtain

$$\mathcal{F}[\,A_h^S f(x)](\omega) = \int_a^b f(x)dx + \frac{1}{4}f(b)h + \frac{2+\pi^2}{96}f'(b)h^2 - \frac{\pi^2}{384}f''(b)h^3 + O(h^4).$$

In Theorem 1 we show that the fourth-order expansion formula of quadrature formula $A_h^S f(x)$ at the left endpoint $x = a$ is given by the Euler-Mclaurin formula.

Claim 1 *Let* $n \geq 10$. *Then*

$$3^{n+1} > n^5. \tag{2.8}$$

Proof. Let $F(n) = (n+1)\ln 3 - 5\ln n$. The first derivative $F'(n) = \ln 3 - 5/n$ is positive for $n > 5/\ln 3$. We have that $F(10) > 0$. When $n \geq 10$ we obtain

$$(n+1)\ln 3 - 5\ln n > 0, \quad 3^{n+1} > n^5.$$

Theorem 1 *Let $f \in C^4[a,b]$. Then*

$$\frac{h}{2}\sum_{k=0}^{n-1}\frac{(-1)^k E_{2k}}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1} f(b-kh) = \int_a^b f(x)dx + \left(\frac{1}{4}f(b) - \frac{1}{2}f(a)\right)h + \left(\frac{2+\pi^2}{96}f'(b) - \frac{1}{12}f'(a)\right)h^2 - \frac{\pi^2}{384}f''(b)h^3 + O(h^4). \quad (2.9)$$

Proof. The Euler numbers satisfy (Borwein et al. 1989)

$$\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)^{2k+1}} = \frac{(-1)^k E_{2k}\pi^{2k+1}}{2^{2k+2}(2k)!}. \quad (2.10)$$

Then

$$1 - \frac{1}{3^{k+1}} < \frac{\bar{E}_k}{2} < 1 - \frac{1}{3^{k+1}} + \frac{1}{5^{k+1}},$$
$$0 < \frac{1}{3^{k+1}} - \frac{1}{5^{k+1}} < 1 - \frac{\bar{E}_k}{2} < \frac{1}{3^{k+1}}.$$

Let $m = \lceil n/2 \rceil$ and $A'_h f(x)$ be the Riemann sum approximation for the definite integral on the interval $[a, b/2]$.

$$A'_h f(x) = h\sum_{k=m}^{n-1} f(b-kh).$$

From the Euler-Mclaurin formula, the fourth-order expansion formula of $A'_h f(x)$ at the left endpoint is

$$L_4(A'_h f(x)) = -\frac{f(a)}{2}h - \frac{f'(a)}{12}h^2 + O(h^4).$$

Denote by $A''_h f(x)$ the approximation for the definite integral on the interval $[a, b/2]$, obtained from (2.6) with coefficients $\bar{E}_k/2$, where $m \leq k \leq n-1$.

$$A''_h f(x) = \frac{h}{2}\sum_{k=m}^{n-1}\bar{E}_k f(b-kh) = \frac{h}{2}\sum_{k=m}^{n-1}\frac{(-1)^k E_{2k}}{(2k)!}\left(\frac{\pi}{2}\right)^{2k+1} f(b-kh).$$

Let $M = \max_{x\in[a,b]}|f(x)|$. Now we estimate $|A'_h f(x) - A''_h f(x)|$.

$$A'_h f(x) - A''_h f(x) = h\sum_{k=m}^{n-1}\left(1 - \frac{\bar{E}_k}{2}\right)f(b-kh),$$

$$|A'_h f(x) - A''_h f(x)| = h\sum_{k=m}^{n-1}\left(1 - \frac{\bar{E}_k}{2}\right)|f(b-kh)| < Mh\sum_{k=m}^{n-1}\left(1 - \frac{\bar{E}_k}{2}\right).$$

From (2.11)

$$0 < 1 - \frac{\bar{E}_k}{2} < \frac{1}{3^{k+1}} \leq \frac{1}{3^{m+1}}, \qquad (m \leq k \leq n-1).$$

Hence

$$|A'_h f(x) - A''_h f(x)| < \frac{Mhm}{3^{m+1}} < \frac{Mhm}{m^5} \leq \frac{16Mh}{n^4} = \frac{16Mh(b-a)^4}{(b-a)^4 n^4} = \frac{16Mh^5}{(b-a)^4}.$$

Therefore, the fourth-order left endpoint expansions of $A'_h f(x)$ and $A''_h f(x)$ satisfy

$$L_4(A'_h f(x)) = L_4(A''_h f(x)) = -\frac{f(a)}{2}h - \frac{f'(a)}{12}h^2 + O(h^4).$$

In Lemma 1 we show that the coefficients of the fourth-order expansion formula at the point $x = b$ of the approximation with generating function $G_1(x)$ are equal to the coefficients of the McLaurin expansion of the function $H_1(x)$. In Theorem 1 we prove that the fourth-order expansion formula of $A_h^S f(x)$ at the left-endpoint is equal to the Euler-McLaurin formula. The results of Lemma 1 and Theorem 1 can be generalized to higher order expansion formulas. We have that $E_0 = \pi/2$. From (2.9) we obtain the second-order quadrature formula

$$A_h^S f(x) = \frac{h}{2}\left(f(a) + \sum_{k=1}^{n-1} \bar{E}_k f(b-kh) + \frac{\pi - 1}{2} f(b) \right) = \int_a^b f(x)dx + O(h^2), \qquad (2.12)$$

The sequence of numbers $\{\bar{E}_k\}_{k=1}^{\infty}$ is increasing and converges to 2. The substitution $\bar{f}(x) = f(a+b-x)$ yields the second order quadrature formula

$$A_h^S \bar{f}(x) = \frac{h}{2}\left(\frac{\pi - 1}{2} \bar{f}(a) + \sum_{k=1}^{n-1} \bar{E}_k \bar{f}(a+kh) + \bar{f}(b) \right) = \int_a^b \bar{f}(x)dx + O(h^2). \qquad (2.13)$$

In Table 1 we compute the error and the order of approximations (2.12) and (2.13) for the definite integral of the functions $\cos x$, $\ln(x+1)$ and $\arctan x$. In Theorem 2 we derive the condition for the integrand function such that the errors of second-order approximations (2.12) and (2.13) are smaller than the error of the midpoint approximation. The midpoint approximation has a fourth-order expansion (1.5)

$$A_M^h f(x) = \int_a^b f(x)dx - \frac{1}{24}\big(f'(b) - f'(a)\big)h^2 + O(h^4). \qquad (2.14)$$

The first and second derivatives of the function $\bar{f}(x)$ satisfy

$$\bar{f}'(a) = -f'(b), \quad \bar{f}'(b) = -f'(a), \quad \bar{f}''(a) = f''(b), \quad \bar{f}''(b) = f''(a). \qquad (2.15)$$

From (2.9) and (2.15) we obtain the third-order expansions of (2.12) and (2.13)

$$A_h^S f(x) = \int_a^b f(x)dx + \left(\frac{2+\pi^2}{96} f'(b) - \frac{1}{12} f'(a) \right) h^2 + O(h^3), \qquad (2.16)$$

$$\bar{A}_h^S f(x) = \int_a^b f(x)dx + \left(\frac{1}{12} f'(b) - \frac{2+\pi^2}{96} f'(a) \right) h^2 + O(h^3). \qquad (2.17)$$

Theorem 2 *Let $f \in C^2[a,b]$. Then*

(i) The error of approximation (2.12) *is smaller than the error of the midpoint approximation when the integrand function $f(x)$ satisfies*

$$\frac{f'(b)}{f'(a)} \in \left(\frac{4}{\pi^2-2}, \frac{12}{\pi^2+6}\right) = (0.5683, 0.7562). \quad (2.18)$$

(ii) The error of approximation (2.13) *is smaller than the error of the midpoint approximation when the integrand function* $f(x)$ *satisfies*

$$\frac{f'(b)}{f'(a)} \in \left(\frac{\pi^2+6}{12}, \frac{\pi^2-2}{4}\right) = (1.3225, 1.9674). \quad (2.19)$$

Proof. From (2.14) and (2.16) the error of approximation (2.12) is smaller than the error of the midpoint approximation when

$$\left|\frac{2+\pi^2}{96} f'(b) - \frac{1}{12} f'(a)\right| < \frac{1}{24} |f'(b) - f'(a)|. \quad (2.20)$$

Denote $R = f'(b)/f'(a)$.

$$\left|\frac{2+\pi^2}{4} R - 2\right| < |R - 1|.$$

The two equations $R - 1 = \pm\left((2+\pi^2)R/4 - 2\right)$ have solutions $R' = 4/(\pi^2-2)$ and $R'' = 12/(\pi^2+6)$. Therefore the error of approximation $A_h^S f(x)$ is smaller than the error of the midpoint approximation when the function $f(x)$ satisfies the condition

$$\frac{f'(b)}{f'(a)} \in \left(\frac{4}{\pi^2-2}, \frac{12}{\pi^2+6}\right).$$

The error terms of (2.16) and (2.17) contain third-order terms (h^3) . Inequality (2.20) is satisfied when the step size h is a small positive number. Similarly we show that, the error of approximation $\bar{A}_h^S f(x)$ is smaller than the error of the midpoint approximation when the function $f(x)$ satisfies (2.19).

Let $f_1(x) = e^{-x/3}$ and $f_2(x) = e^{x/2}$ on the interval [0,1]. The functions $f_1(x)$ and $f_2(x)$ satisfy conditions (2.18) and (2.19), because

$$\frac{f_1'(1)}{f_1'(0)} = e^{-1/3} = 0.7165, \quad \frac{f_2'(1)}{f_2'(0)} = e^{1/2} = 1.6487.$$

In Table 2 and Table 3 we compare the error and the order of approximations (2.12) and (2.13) with the error and the order of the midpoint approximation of the definite integral of the functions $e^{-x/3}$ and $e^{x/2}$ on the interval [0,1].

Table 1.

Error and order of second-order approximation (2.12) for the definite integral of the functions $f(x) = \cos x$ on the interval [1,3], $f(x) = \ln(x+1)$ on [0,2] and approximation (2.13) for $f(x) = \arctan x$ on [0,3].

h	$\cos x$		$\ln(x+1)$		$\arctan x$	
	Error	**Order**	**Error**	**Order**	**Error**	**Order**
0.025	0.000026603	2.0083	0.00002628	1.9973	0.00007206	1.9995
0.0125	6.63×10^{-6}	2.0044	6.58×10^{-6}	1.9987	0.00001802	1.9998
0.00625	1.66×10^{-6}	2.0022	1.64×10^{-6}	1.9994	4.50×10^{-6}	1.9999
0.003125	4.13×10^{-6}	2.0011	4.11×10^{-7}	1.9997	1.13×10^{-6}	2.0000

Source: Own calculations

Table 2.

Error and order of approximation (2.12)[left] and the midpoint approximation (1.3)[right] for the definite integral of $f(x) = e^{-x/3}$ on [0,1].

h	**Error**	**Order**
0.025	1.13×10^{-6}	2.0233
0.0125	2.78×10^{-7}	2.0120
0.00625	6.90×10^{-8}	2.0060
0.003125	1.72×10^{-8}	2.0030

h	**Error**	**Order**
0.025	2.46×10^{-6}	1.9999
0.0125	6.15×10^{-7}	2.0000
0.00625	1.54×10^{-7}	2.0000
0.003125	3.84×10^{-8}	2.0000

Source: Own calculations

Table 3.

Error and order of approximation (2.13)[left] and the midpoint approximation (1.3)[right] for the definite integral of $f(x) = e^{x/2}$ on [0,1].

h	**Error**	**Order**
0.025	4.20×10^{-6}	1.9643
0.0125	1.07×10^{-6}	1.9827
0.00625	2.67×10^{-7}	1.9915
0.003125	6.69×10^{-8}	1.9958

h	**Error**	**Order**
0.025	8.45×10^{-6}	1.9999
0.0125	2.11×10^{-6}	2.0000
0.00625	5.28×10^{-7}	2.0000
0.003125	1.32×10^{-7}	2.0000

Source: Own calculations

3. Second-order quadrature formulas with generating function $G_2(x)$

In this section we use the McLaurin series expansion of tangent to derive the fourth-order expansion formula (3.3) of the approximation for the definite integral with generating function $G_2(x) = \pi\tan(\pi\sqrt{x}/2)/(4\sqrt{x})$ and the second-order quadrature formulas (3.6) and (3.7). From (1.7)

$$\tan\left(\frac{\pi\sqrt{x}}{2}\right) = \sum_{k=1}^{\infty} \frac{(-4)^k(1-4^k)B_{2k}}{(2k)!}\left(\frac{\pi}{2}\right)^{2k-1} x^{(2k-1)/2}, \qquad (|x|<1).$$

The function $G_2(x)$ has a McLaurin series expansion

$$G_2(x) = \frac{\pi}{4\sqrt{x}}\tan\left(\frac{\pi\sqrt{x}}{2}\right) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-4)^{k+1}(1-4^{k+1})B_{2k+2}}{(2k+2)!}\left(\frac{\pi}{2}\right)^{2k+2} x^k,$$

$$G_2(x) = \frac{1}{2}\sum_{k=0}^{\infty} \bar{B}_k x^k = \frac{1}{2}\sum_{k=0}^{\infty} \frac{(-1)^k(4^{k+1}-1)\pi^{2k+2}B_{2k+2}}{(2k+2)!}x^k. \qquad (3.1)$$

Let

$$H_2(x) = xG_2(e^{-x}) = \frac{\pi}{4}xe^{x/2}\tan\left(\frac{\pi e^{-x/2}}{2}\right).$$

Similarly to section 2, we determine the values of the derivatives of the function $H_2(x)$ at $x = 0$.

$$H_2(0) = 1, \qquad H_2'(0) = \frac{3}{4}, \qquad H_2''(0) = \frac{13-\pi^2}{24}, \qquad H_2'''(0) = \frac{12-\pi^2}{32}.$$

The function $H_2(x)$ has a fourth-order McLaurin series

$$H_2(x) = 1 + \frac{3}{4}x + \frac{13-\pi^2}{48}x^2 + \frac{12-\pi^2}{192}x^3 + O(x^4).$$

Denote by

$$A_h^T f(x) = \frac{h}{2}\sum_{k=0}^{\infty} \bar{B}_k f(b-kh),$$

the approximation for the definite integral with a generating function $G_2(x)$. In Lemma 2 and Theorem 3 we obtain the fourth-order expansion formula of approximation$A_h^T f(x)$.

Lemma 2 *Let* $f \in C^4[a,b]$ *and* $f^{(n)}(0) = 0$ *for* $n = 0,1,2,3$. *Then*

$$A_h^T f(x) = \frac{h}{2}\sum_{k=0}^{n-1} \frac{(-1)^k(4^{k+1}-1)\pi^{2k+2}B_{2k+2}}{(2k+2)!} f(b-kh) = \int_a^b f(x)dx + \frac{3}{4}f(b)h + \frac{13-\pi^2}{48}f'(b)h^2 + \frac{12-\pi^2}{192}f''(b)h^3 + O(h^4). \quad (3.2)$$

Theorem 3 *Let* $f \in C^4[a,b]$. *Then*

$$A_h^T f(x) = \frac{h}{2}\sum_{k=0}^{n-1} \bar{B}_k f(b-kh) = \int_a^b f(x)dx + \left(\frac{3}{4}f(b) - \frac{1}{2}f(a)\right)h + \left(\frac{13-\pi^2}{48}f'(b) - \frac{1}{12}f'(a)\right)h^2 + \frac{12-\pi^2}{192}f''(b)h^3 + O(h^4). \quad (3.3)$$

The Bernoulli numbers are expressed with values of the zeta function as

$$B_{2k+2} = 2(-1)^k\zeta(2k+2)\frac{(2k+2)!}{(2\pi)^{2k+2}}. \quad (3.4)$$

Then

$$\bar{B}_k = \frac{(-1)^k(4^{k+1}-1)\pi^{2k+2}B_{2k+2}}{(2k+2)!} = 2\frac{4^{k+1}-1}{4^{k+1}}\zeta(2k+2),$$

$$\frac{\bar{B}_k}{2} = \left(1 - \frac{1}{4^{k+1}}\right)\zeta(2k+2). \quad (3.5)$$

When $k \geq 1$, the zeta function satisfies (Kouba 2013)

$$\zeta(2k) < 1 + \frac{1}{2^{2k}} + \frac{2}{2k-1}\frac{1}{2^{2k}} < 1 + \frac{3}{4^k}.$$

Hence

$$\frac{\bar{B}_k}{2} = \left(1 - \frac{1}{4^{k+1}}\right)\zeta(2k+2) < \left(1 - \frac{1}{4^{k+1}}\right)\left(1 + \frac{3}{4^{k+1}}\right) < 1 + \frac{2}{4^{k+1}}.$$

The proofs of Lemma 2 and Theorem 3 are similar to the proofs of Lemma 1 and Theorem1. The value of $\bar{B}_0$ is $\bar{B}_0 = \pi^2/4$. From (3.3) we obtain the second-order quadrature formula

$$A_h^T f(x) = \frac{h}{2}\left(f(a) + \sum_{k=0}^{n-1} \bar{B}_k f(b-kh) + \frac{\pi^2-6}{4}f(b)\right) = \int_a^b f(x)dx + O(h^2). \quad (3.6)$$

From (3.5), we can express the formula for approximation $A_h^T f(x)$ as

$$h\left(\frac{f(a)}{2}+\sum_{k=0}^{n-1}\left(1-\frac{1}{4^{k+1}}\right)\zeta(2k+2)f(b-kh)-\frac{3f(b)}{4}\right)=\int_a^b f(x)dx+O(h^2).$$

The substitution $\bar{f}(x)=f(a+b-x)$ yields the second-order quadrature formula

$$\bar{A}_h^T f(x)=\frac{h}{2}\left(\frac{\pi^2-6}{4}f(a)+\sum_{k=0}^{n-1}\bar{B}_k f(a+kh)+f(b)\right)=\int_a^b f(x)dx+O(h^2). \quad (3.7)$$

In Table 4 we compute the error and the order of approximations (3.6) and (3.7) for the definite integral of the functions $\cos x, \ln(x+1)$ and $\arctan x$. In Theorem 4 we determine the conditions for the function $f(x)$ which ensure that the error of quadrature formulas (3.6) and (3.7) is smaller than the error of the midpoint approximation. From (3.3) approximations $A_h^T f(x)$ and $\bar{A}_h^T f(x)$ have third-order expansion formulas

$$A_h^T f(x)=\int_a^b f(x)dx+\left(\frac{13-\pi^2}{48}f'(b)-\frac{1}{12}f'(a)\right)h^2+O(h^2), \quad (3.8)$$

$$\bar{A}_h^T f(x)=\int_a^b f(x)dx+\left(\frac{1}{12}f'(b)-\frac{13-\pi^2}{48}f'(a)\right)h^2+O(h^2). \quad (3.9)$$

Theorem 4 *Let $f\in C^2[a,b]$. Then*

(i) The error of approximation (3.6) *is smaller than the error of the midpoint approximation when the integrand function $f(x)$ satisfies*

$$\frac{f'(b)}{f'(a)}\in\left(\frac{6}{15-\pi^2},\frac{2}{11-\pi^2}\right)=(1.1695,1.7693). \quad (3.10)$$

(ii) The error of approximation (3.7) *is smaller than the error of the midpoint approximation when the integrand function $f(x)$ satisfies*

$$\frac{f'(b)}{f'(a)}\in\left(\frac{11-\pi^2}{2},\frac{15-\pi^2}{6}\right)=(0.5652,0.855). \quad (3.11)$$

Proof. From (2.14) and (3.9) the error of approximation $\bar{A}_h^T f(x)$ is smaller than the error of the midpoint approximation when

$$\left|\frac{1}{12}f'(b)-\frac{13-\pi^2}{48}f'(a)\right|<\frac{1}{24}|f'(b)-f'(a)|.$$

Denote $R=f'(b)/f'(a)$. Then $2|R+(\pi^2-13)/4|<|R-1|$. The equations $1-R=\pm(R+(\pi^2-13)/4)$ have solutions: $R'=(11-\pi^2)/2$ and $R''=(15-\pi^2)/6$. The error of approximation (3.7) is smaller than the error of the midpoint approximation when the function $f(x)$ satisfies the condition

$$\frac{f'(b)}{f'(a)}\in\left(\frac{11-\pi^2}{2},\frac{15-\pi^2}{6}\right).$$

Similarly, the error of approximation (3.6) is smaller than the error of the midpoint approximation when the function $f(x)$ satisfies (3.11).

The functions $f(x)=e^{x/2}$ and $f(x)=e^{-x/3}$ satisfy conditions (3.10) and (3.11) on the interval $[0,1]$. In Table 3.2 and Table 3.3 we compare approximations (3.6) and (3.7) with the midpoint approximation of the functions $e^{-x/3}$ and $e^{x/2}$ on the interval $[0,1]$.

Table 4.

Error and order of second-order approximation (3.6) for the definite integral of the functions $f(x) = \cos x$ on the interval [1,3], $f(x) = \ln(x+1)$ on [0,2] and approximation (2.13) for $f(x) = \arctan x$ on [0,3].

h	$\cos x$		$\ln(x+1)$		$\arctan x$	
	Error	**Order**	**Error**	**Order**	**Error**	**Order**
0.025	0.000038246	2.0065	0.00002628	1.9973	0.00003556	2.0006
0.0125	9.54×10^{-6}	2.0032	6.58×10^{-6}	1.9987	8.89×10^{-6}	2.0002
0.00625	2.38×10^{-6}	2.0016	1.64×10^{-6}	1.9994	2.22×10^{-6}	2.0001
0.003125	5.95×10^{-7}	2.0008	4.11×10^{-7}	1.9997	5.56×10^{-7}	2.0000

Source: Own calculations

Table 5.

Error and order of approximation (3.6)[left] and the midpoint approximation (1.3)[right] for the definite integral of $f(x) = e^{x/2}$ on [0,1].

h	**Error**	**Order**
0.025	7.63×10^{-6}	1.9643
0.0125	1.90×10^{-6}	1.9827
0.00625	4.73×10^{-7}	1.9915
0.003125	1.18×10^{-7}	1.9958

h	**Error**	**Order**
0.025	8.45×10^{-6}	1.9999
0.0125	2.11×10^{-6}	2.0000
0.00625	5.28×10^{-7}	2.0000
0.003125	1.32×10^{-7}	2.0000

Source: Own calculations

Table 6.

Error and order of approximation (3.7)[left] and the midpoint approximation (1.3)[right] for the definite integral of $f(x) = e^{-x/3}$ on [0,1].

h	**Error**	**Order**
0.025	1.17×10^{-6}	2.0233
0.0125	2.89×10^{-7}	2.0120
0.00625	7.20×10^{-8}	2.0060
0.003125	1.79×10^{-8}	2.0030

h	**Error**	**Order**
0.025	2.46×10^{-6}	1.9999
0.0125	6.15×10^{-7}	2.0000
0.00625	1.54×10^{-7}	2.0000
0.003125	3.84×10^{-8}	2.0000

Source: Own calculations

4. Higher-order quadrature formulas

In section 2 and section 3 we obtained second-order quadrature formulas (2.12), (2.13), (3.6) and (3.7). The trapezoidal approximation has a fourth-order expansion formula (2.14). From Theorem 1, Theorem 3 and (2.15), approximations (2.13) and (3.7) have fourth-order expansion formulas

$$\bar{A}_h^S f(x) = \frac{h}{2}\left(\frac{\pi-1}{2}f(a) + \sum_{k=1}^{n-1}\bar{E}_k f(a+kh) + f(b)\right) = \int_a^b f(x)dx - \frac{2+\pi^2}{96}f'(a)h^2 - \frac{\pi^2}{384}f''(a)h^3 + \frac{1}{12}f'(b)h^2 + O(h^4),$$

$$\bar{A}_h^T f(x) = \frac{h}{2}\left(\frac{\pi^2-6}{4}f(a) + \sum_{k=1}^{n-1}\bar{B}_k f(a+kh) + f(b)\right) = \int_a^b f(x)dx - \frac{13-\pi^2}{48}f'(a)h^2 + \frac{12-\pi^2}{192}f''(a)h^3 + \frac{1}{12}f'(b)h^2 + O(h^4).$$

By substituting $K = n-k$ in (2.12) and (3.6) we obtain

$$A_h^S f(x) = \frac{h}{2}\left(f(a) + \sum_{K=1}^{n-1}\bar{E}_k f(a+Kh) + \frac{\pi-1}{2}f(b)\right) = \int_a^b f(x)dx - \frac{1}{12}f'(a)h^2 + \frac{2+\pi^2}{96}f'(b)h^2 - \frac{\pi^2}{384}f''(b)h^3 + O(h^4),$$

$$A_h^T f(x) = \frac{h}{2}\left(f(a) + \sum_{K=1}^{n-1}\bar{B}_k f(a+Kh) + \frac{\pi-1}{2}f(b)\right) = \int_a^b f(x)dx - \frac{1}{12}f'(a)h^2 + \frac{13-\pi^2}{48}f'(b)h^2 + \frac{12-\pi^2}{192}f''(b)h^3 + O(h^4).$$

4.1 Third-order quadrature formulas

Similarly to the construction of the Simpson's approximation we construct third and fourth order approximations (4.2), (4.3) and (4.4) for the definite integral as linear combinations of the trapezoidal approximation (1.2) and approximations (2.12), (2.13), (3.6) and (3.7). Let

$$B_3^h f(x) = c_0 A_T^h f(x) + c_1 A_h^S f(x) + c_2 \bar{A}_h^S f(x),$$

where, the numbers c_i satisfy the system of equations

$$\begin{cases} c_0 + c_1 + c_2 = 1, \\ 8c_0 + (2+\pi^2)c_1 + 8c_2 = 0, \\ 8c_0 + 8c_1 + (2+\pi^2)c_2 = 0. \end{cases} \quad (4.1)$$

The condition $c_0 + c_1 + c_2 = 1$ for the coefficients c_i ensures that $B_3^h f(x)$ is an approximation for the definite integral. The second and third equations of (4.1) are chose such that the second order terms of $B_3^h f(x)$ at the endpoints $x = a$ and $x = b$ are equal to zero. The system of equations (4.1) has the solution

$$c_0 = \frac{10+\pi^2}{\pi^2-6}, c_1 = c_2 = \frac{8}{6-\pi^2}.$$

With this choice of the coefficients c_i, approximation $B_3^h f(x)$ for the definite integral has third-order accuracy

$$B_3^h f(x) = \frac{h}{\pi^2-6}\left(w_0(f(a)+f(b)) + \sum_{k=1}^{n-1} w_k f(a+kh)\right) = \int_a^b f(x)dx + O(h^3). \quad (4.2)$$

The weights of approximation $B_3^h f(x)$ satisfy

$$w_0 = \frac{1}{2}(\pi^2 - 4\pi + 6), \quad w_k = \pi^2 + 10 - 4(\bar{E}_k + \bar{E}_{n-k}), \quad (1 \le k \le n-k).$$

Similarly, we construct a third-order approximation $\bar{B}_3^h f(x)$ for the definite integral as a linear combination of the trapezoidal approximation and approximations (3.6) and (3.7). Let

$$\bar{B}_3^h f(x) = c_0 A_T^h f(x) + c_1 A_h^T f(x) + c_2 \bar{A}_h^T f(x),$$

where, the numbers c_i satisfy

$$\begin{cases} c_0 + c_1 + c_2 = 1, \\ 4c_0 + (13 - \pi^2)c_1 + 4c_2 = 0, \\ 4c_0 + 4c_1 + (13 - \pi^2)c_2 = 0. \end{cases}$$

The system of equations has the solution

$$c_0 = \frac{\pi^2 - 17}{\pi^2 - 9}, c_1 = c_2 = \frac{4}{\pi^2 - 9}.$$

We obtain the third-order approximation $\bar{B}_3^h f(x)$ for the definite integral

$$\bar{B}_3^h f(x) = \frac{h}{\pi^2 - 9}\left(w_0(f(a) + f(b)) + \sum_{k=1}^{n-1} w_k f(a + kh) \right) = \int_a^b f(x)dx + O(h^3). \quad (4.3)$$

$$w_0 = \frac{1}{2}(2\pi^2 - 19), \quad w_k = \pi^2 - 17 + 2(\bar{B}_k + \bar{B}_{n-k}), \quad (1 \le k \le n - k).$$

In Table 7 and Table 8 we compute the error and the order of approximations (4.2) and (4.3) for the definite integral of the functions $\cos x, \ln(x + 1)$ and $\arctan x$.

4.2 Fourth-order quadrature formula

We construct a fourth-order quadrature formula as a linear combination of the trapezoidal approximation and approximations (2.12), (2.13), (3.6), (3.7). Let

$$B_4^h f(x) = c_0 A_T^h f(x) + c_1 A_h^S f(x) + c_2 \bar{A}_h^S f(x) + c_3 A_h^T f(x) + c_4 \bar{A}_h^T f(x).$$

Approximation $B_4^h f(x)$ has a fourth-order accuracy when the coefficients c_i satisfy

$$\begin{cases} c_0 + c_1 + c_2 + c_3 + c_4 = 1, \\ 8c_0 + 8c_1 + (2 + \pi^2)c_2 + 8c_3 + 2(13 - \pi^2)c_4 = 0, \\ 8c_0 + (2 + \pi^2)c_1 + 8c_2 + 4(13 - \pi^2)c_3 + 8c_4 = 0, \\ \pi^2 c_2 + 2(\pi^2 - 12)c_4 = 0, \\ \pi^2 c_1 + 2(\pi^2 - 12)c_3 = 0. \end{cases}$$

Let $D = 2\pi^4 - 27\pi^2 + 72$. The system of equations has the solution

$$c_0 = \frac{2\pi^4 - 19\pi^2 - 120}{D}, c_1 = c_2 = \frac{4(12 - \pi^2)}{D}, c_3 = c_4 = \frac{4\pi^2}{D}.$$

With this choice of the coefficients c_i we obtain the fourth-order approximation for the definite integral

$$B_4^h f(x) = \frac{h}{D}\left(w_0(f(a) + f(b)) + \sum_{k=1}^{n-1} w_k f(a + kh) \right) = \int_a^b f(x)dx + O(h^4). \quad (4.4)$$

where $w_0 = (3\pi^4 - 4\pi^3 - 25\pi^2 + 48\pi - 72)/2$ and

$$w_k = 2\pi^4 - 19\pi^2 - 120 + 4(12 - \pi^2)(\bar{E}_k + \bar{E}_{n-k}) + 2\pi^2(\bar{E}_k + \bar{E}_{n-k}),$$

for $1 \le k \le n - k$. In Table 9 we compute the error and the order of approximation (4.4) for the definite integral of the functions $\cos x, \ln(x + 1)$ and $\arctan x$.

Table 7.

Error and order of third-order approximation (4.2) for the definite integral of the functions $f(x) = \cos x, \ln(x+1), \arctan x$ on the interval [10,30].

h	$\cos x$		$\ln(x+1)$		$\arctan x$	
	Error	**Order**	**Error**	**Order**	**Error**	**Order**
0.025	0.00060862	3.0494	7.57×10^{-6}	2.9722	1.63×10^{-6}	2.9498
0.0125	0.00007386	3.0427	6.58×10^{-7}	2.9857	2.07×10^{-7}	2.9740
0.00625	9.07×10^{-6}	3.0261	1.64×10^{-7}	2.9928	2.61×10^{-8}	2.9868
0.003125	1.22×10^{-6}	3.0143	4.11×10^{-8}	2.9969	3.28×10^{-9}	2.9945

Source: Own calculations

Table 8.

Error and order of third-order approximation (4.3) for the definite integral of the functions $f(x) = \cos x, \ln(x+1), \arctan x$ on the interval [10,30].

h	$\cos x$		$\ln(x+1)$		$\arctan x$	
	Error	**Order**	**Error**	**Order**	**Error**	**Order**
0.025	0.00058322	3.0509	7.28×10^{-6}	2.9735	1.57×10^{-6}	2.9520
0.0125	0.00007083	3.0416	9.19×10^{-6}	2.9864	1.99×10^{-7}	2.9752
0.00625	8.70×10^{-6}	3.0251	1.15×10^{-6}	2.9930	2.51×10^{-8}	2.9872
0.003125	1.08×10^{-6}	3.0137	1.45×10^{-7}	2.9953	3.16×10^{-9}	2.9909

Source: Own calculations

Table 9.

Error and order of third-order approximation (4.4) for the definite integral of the functions $f(x) = \cos x, \ln(x+1), \arctan x$ on the interval [0,30].

h	$\cos x$		$\ln(x+1)$		$\arctan x$	
	Error	**Order**	**Error**	**Order**	**Error**	**Order**
0.025	0.00016129	4.1502	0.00014554	3.3684	0.00017815	2.7198
0.0125	9.20×10^{-6}	4.1317	0.00001189	3.6129	0.00001474	3.5951
0.00625	5.43×10^{-7}	4.0828	1.64×10^{-7}	3.7781	9.99×10^{-7}	3.8829
0.003125	3.29×10^{-8}	4.0469	4.11×10^{-8}	3.8583	6.33×10^{-8}	3.9805

Source: Own calculations

5. Conclusion

In the present paper we obtained second, third and fourth order quadrature formulas based on the expansion formulas of the approximations for the definite integral with generating functions $\pi\sec(\pi\sqrt{x}/2)/4$ and $\pi\tan(\pi\sqrt{x}/2)/(4\sqrt{x})$. In Theorem 2 and Theorem 4 we derive the conditions for the integrand function which ensure that the error of second-order quadrature formulas (2.12), (2.13), (3.6) and (3.7) is smaller than the error of the midpoint rule. In future work we are going to extend the results of this paper to multidimensional and fractional integrals.

Acknowledgements

This work was supported by the Bulgarian Academy of Sciences through the Program for Career Development of Young Scientists, Grant DFNP-17-88/2017, Project "Efficient Numerical Methods with an Improved Rate of Convergence for Applied Computational Problems" and by the Bulgarian National Science Fund under Grant DFNI I02-20/2014, Project "Efficient Parallel Algorithms for Large-Scale Computational Problems".

Literature

Abramowitz, M. & Stegun I. A., 1964. Handbook of mathematical functions with formulas, graphs, and mathematical tables. *Dover, New York*.

Atanassov, E. & Dimov, I., 1999. A new optimal Monte Carlo method for calculating integrals of smooth functions. *Monte Carlo Methods and Applications 5(2*), pp. 149-167.

Atkinson, M. D., 1986. How to compute the series expansions of secx and tanx. *Amer. Math. Monthly 93*, pp. 387-388.

Borwein, J. M., Borwein, P. B. & Dilcher, D., 1989. Pi, Euler numbers and asymptotic expansions. *The American Mathematical Monthly 96(8)*, pp. 681-687.

Brent, R. P. & Harvey, D., 2013. Fast computation of Bernoulli, tangent and secant numbers. *Proceedings of amWorkshop on Computational and Analytical Mathematics in honour of Jonathan Borwein's 60th birthday, Springer Proceedings in Mathematics and Statistics, 50*, pp. 127-142.

Chen, M. & Deng, W., 2016. A second-order accurate numerical method for the space-time tempered fractional diffusion-wave equation. *arXiv:1610.02661*.

Davis, P. J. & Rabinowitz, P., 1984. Methods of numerical integration: Second edition, *Academic Press.*

Dimitrov, Y., 2014. Numerical approximations for fractional differential equations. *Journal of Fractional Calculus and Applications, 5(3S), No. 22*, pp. 1-45.

Dimitrov, Y., 2016. Second-order approximation for the Caputo derivative. *Journal of Fractional Calculus and Applications, 7(2)*, pp. 175-195.

Dimitrov, Y., 2016. Higher-order numerical solutions of the fractional relaxation-oscillation equation using fractional integration. *arXiv:1603.08733*.

Dimov, I., 2008. Monte Carlo methods for applied scientists. *World Scientific.*

Ding, H. & Li, C., 2016. High-order numerical algorithms for Riesz derivatives via constructing new generating functions. *Journal of Scientific Computing.*

Ding, H. & Li, C., 2016. High-order algorithms for Riesz derivative and their applications (III). *Fract. Calc. Appl. Anal. 19(1),* 19–55 (2016)

Georgieva, R., 2009. Computational complexity of Monte Carlo algorithms for multidimensional integrals and integral equations. *Ph.D. thesis, BAS*.

Gil, A., Segura, J. & Temme, N. M., 2007. Numerical methods for special functions. *Society for Industrial and Applied Mathematics (SIAM)*, Philadelphia, PA.

Knuth, D. E. & Buckholtz, T. J., 1967. Computation of tangent, Euler and Bernoulli numbers. *Math. Comput. 21*, pp. 663–688.

Kouba O., 2013. Bernoulli polynomials and applications. *Lecture Notes.*

Lubich, C., 1986. Discretized fractional calculus. *SIAM J. Math. Anal. 17*, pp. 704-719.

Sahakyan, K., 2014. New quadrature and cubature formulas of Hermite type. *Universal Journal of Applied Mathematics 2(9)*, pp. 299-304.

Tadjeran, C., Meerschaert, M. M. & Scheffer, H. P., 2006. A second-order accurate numerical approximation for the fractional diffusion equation. *Journal of Computational Physics 213*, pp. 205-213 (2006)

Tian, W. Y., Zhou, H. & Deng, W.H., 2015. A class of second order difference approximations for solving space fractional diffusion equations. *Math. Comput. 84*, 1703-1727.

Todorov, V. & Dimov, I., 2016. Monte Carlo methods for multidimensional integration for European option pricing. AIP Conf. Proc. 1773, 100009Atkinson, M. D., 1986.

Tuan, V. K. & Gorenflo, R., 1995. Extrapolation to the limit for numerical fractional differentiation. Z. Agnew. Math. Mech. 75, pp. 646-648. doi:10.1002/19950750826.